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EFFECTIVE DOWNSTREAM BOUNDARY CONDITIONS FOR INCOMPRESSIBLE NAVIER–STOKES EQUATIONS

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SUMMARY

The aim of this paper is to give open boundary conditions for the incompressible Navier–Stokes equations. From a weak formulation in velocity–pressure variables, some natural boundary conditions involving the traction or pseudotraction and inertial terms are established. Numerical experiments on the flow behind a cylinder show the efficiency of these conditions, which convey properly the vortices downstream. Comparisons with other boundary conditions for the velocity and pressure are also performed.

KEY WORDS Boundary conditions Artificial boundary Internal flow Incompressible flows

1. INTRODUCTION

The motivation of this work is to establish boundary conditions on open domains for the incompressible Navier–Stokes equations in order to perform long-time simulations at high Reynolds numbers.

There are so many papers in the literature about the incompressible Navier–Stokes equations that the reader can be overwhelmed. In particular, various authors propose numerous formulations and open boundary conditions. For an exhaustive review we refer the reader to Reference 1. Here we limit the study to the velocity–pressure equations which can be used in 2D as well as in 3D without any changes. However, several weak formulations have been built, most of them giving specific boundary conditions on downstream artificial boundaries.

For example, more than 10 years ago Peyret and Rebourcet[2] gave a homogeneous Neumann condition for the tangential velocity component and a condition for the pressure through a parabolization of the momentum equation, both coupled with the continuity equation. For low Reynolds numbers these conditions are numerically stable and yield good solutions. On the other hand, they give rise to strong reflections at high Reynolds numbers.

Later in the 1980s Pironneau[3] and Bègue *et al.*[4,5] proposed a Dirichlet condition on the pressure for the Stokes model and gave a Bernoulli equation for the Navier–Stokes model. In both cases the tangential component of the velocity is set equal to zero.

At the same time Halpern[6] and Halpern and Schatzman[7] studied artificial boundary conditions for the linear advection–diffusion equation and the linearized Navier–Stokes equations, following the theory of absorbing boundary conditions developed for the wave equation.

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Our work is closer to the studies carried out by Conca,[8] Ganesh,[9] Gresho[1,10] and Verfürth[11,12] involving boundary conditions on the traction or pseudotraction, the most famous being to set the pseudotraction equal to zero. Starting from the genuine Navier–Stokes equations, we build a weak formulation on the velocity and pressure, including natural boundary conditions, which leads to a well-posed problem. For Stokes flows these conditions reduce to taking the pseudotraction equal to zero or the traction equal to the traction of a reference flow, for instance Poiseuille flow. Otherwise we take into account the effects of the inertial terms. Numerical experiments on the flow behind a cylinder in a channel show that these conditions are very efficient and do not affect the solution.

2. THE GOVERNING EQUATIONS AND NOTATIONS

The dimensionless Navier–Stokes equations read

$$\partial_t U + (U \cdot \nabla) \cdot U - \operatorname{div} \tilde{\sigma}(U, p) = 0, \tag{1a}$$

$$\operatorname{div} U = 0 \tag{1b}$$

or

$$\partial_t U + (U \cdot \nabla) \cdot U - \operatorname{div} \sigma(U, p) = 0, \tag{2a}$$

$$\operatorname{div} U = 0, \tag{2b}$$

where $U = (u_i)_i$ is the velocity, p is the pressure, Re is the Reynolds number, $\tilde{\sigma}(U, p)$ is the pseudostress tensor and $\sigma(U, p)$ is the stress tensor:

$$\tilde{\sigma}(U, p) = \frac{1}{Re} \nabla U - pI,$$

$$\sigma(U, p) = \frac{2}{Re} D(U) - pI, \quad \text{with } D(U)_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

We solve equations (1) or (2) in a domain $\Omega \subset \mathbb{R}^N$ of boundary $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \varnothing$, associated with the initial condition

$$U(0) = U_0 \quad \text{in } \Omega, \tag{3}$$

a Dirichlet boundary condition on Γ_D,

$$U = U_D \quad \text{on} \quad \Gamma_D \times (0, T), \tag{4}$$

and one of the following Neumann-type boundary conditions on Γ_N:

$$\tilde{\sigma}(U, p) \cdot n - \tfrac{1}{2}\Theta(U \cdot n)(U - V_0) - \tfrac{1}{2}\Theta((U - V_0) \cdot n)V_0 = \tilde{G}_1 \quad \text{on } \Gamma_N \times (0, T), \tag{5a}$$

$$\tilde{\sigma}(U, p) \cdot n - \tfrac{1}{2}\Theta((U - V_0) \cdot n)U = \tilde{G}_2 \quad \text{on } \Gamma_N \times (0, T), \tag{5b}$$

$$\tilde{\sigma}(U, p) \cdot n - \tfrac{1}{2}\Theta(U \cdot n)(U - V_0) = \tilde{G}_3 \quad \text{on } \Gamma_N \times (0, T), \tag{5c}$$

$$\tilde{\sigma}(U, p) \cdot n - \tfrac{1}{2}\Theta((U - V_0) \cdot n)(U - V_0) = \tilde{G}_4 \quad \text{on } \Gamma_N \times (0, T) \tag{5d}$$

or

$$\sigma(U, p)\cdot n - \tfrac{1}{2}\Theta(U\cdot n)(U - V_0) - \tfrac{1}{2}\Theta((U - V_0)\cdot n)V_0 = G_1 \quad \text{on } \Gamma_N \times (0, T), \tag{6a}$$

$$\sigma(U, p)\cdot n - \tfrac{1}{2}\Theta((U - V_0)\cdot n)U = G_2 \quad \text{on } \Gamma_N \times (0, T), \tag{6b}$$

$$\sigma(U, p)\cdot n - \tfrac{1}{2}\Theta(U\cdot n)(U - V_0) = G_3 \quad \text{on } \Gamma_N \times (0, T), \tag{6c}$$

$$\sigma(U, p)\cdot n - \tfrac{1}{2}\Theta((U - V_0)\cdot n)(U - V_0) = G_4 \quad \text{on } \Gamma_N \times (0, T). \tag{6d}$$

where V_0 is a reference velocity field we specify below, n is the unit normal vector pointing out of the domain, $\tilde{G}_k$ and G_k are data to be specified and $\Theta(a)$ is one of the real functions

$$\Theta(a) = a, \qquad \Theta(a) = -a^-, \qquad \Theta(a) = -|a|.$$

with the notation

$$a = a^+ - a^- = 2a^+ - |a|.$$

These boundary conditions are mathematically very close to each other. They are derived from a weak formulation of the Navier–Stokes equations that ensures an energy estimate.[13] The three forms of the Θ-function come from the control of the boundary terms resulting from the integration by parts of the convection terms. More precisely, $\Theta(a) = a$, $\Theta(a) = a^-$ and $\Theta(a) = -|a|$ correspond respectively to eliminating the boundary term on Γ_N or keeping the positive part of the boundary term on Γ_N once or twice. Indeed, keeping this positive part allows us to derive an energy estimate.

3. WEAK FORMULATION

Let us denote by (V_0, p_0) the solution of the Stokes problem in Ω satisfying

$$V_0 = U_D \quad \text{on } \Gamma_D,$$

$$\int_{\Gamma_D} U_D\cdot n\, d\gamma + \int_{\Gamma_D} U_N\cdot n\, d\gamma = 0.$$

For instance we can take for U_N the restriction on Γ_N of a Stokes flow computed on a larger domain.

As an example we introduce the weak formulation corresponding to (6c) for $V = U - V_0$, $q = p - p_0$ and H a given external force on Γ_N: find a couple (V, q) solution of

$$\begin{aligned}
&\int_\Omega \partial_t V\cdot W\, d\omega + \tfrac{1}{2}\int_\Omega \{[(V + V_0)\cdot\nabla V]\cdot W - [(V + V_0)\cdot\nabla W]\cdot V\}\, d\omega \\
&\quad + \int_\Omega (V\cdot\nabla V_0)\cdot W\, d\omega + \int_\Omega (V_0\cdot\nabla V_0)\cdot W\, d\omega + \frac{2}{Re}\int_\Omega D(V):D(W)\, d\omega \\
&\quad - \int_\Omega q \operatorname{div} W\, d\omega + \int_{\Gamma_N} \alpha[(V + V_0)\cdot n]^+ V\cdot W\, d\gamma = \int_{\Gamma_N} H\cdot W\, d\gamma,
\end{aligned}$$

$$\int_\Omega \pi \operatorname{div} V\, d\omega = 0$$

for any couple of test functions (W, π).

In these equations α is a non-negative real number and we adopt the notation

$$\sigma : \sigma' = \sum_{ij} \sigma_{ij}\sigma'_{ij}.$$

To get the first equation, we use a symmetric form of the non-linear term and we add the boundary term to ensure coercivity. For more details see Reference 13. To convince the reader, let us formally interpret this formulation.

First we remark that by symmetry we have

$$D(V):D(W) = D(V):\nabla W$$

and by the Stokes formula

$$\int_\Omega \left(\frac{2}{Re} D(V):\nabla W - q \operatorname{div} W\right) \mathrm{d}\omega = -\int_\Omega \operatorname{div} \sigma(V, q)\cdot W \,\mathrm{d}\omega + \int_\Gamma [\sigma(V, q)\cdot n]\cdot W \,\mathrm{d}\gamma.$$

Moreover, since V and V_0 are divergence-free vector fields, we get the identity

$$\int_\Omega [(V + V_0)\cdot \nabla W]\cdot V \,\mathrm{d}\omega = -\int_\Omega [(V + V_0)\cdot \nabla V]\cdot W \,\mathrm{d}\omega + \int_\Gamma [(V + V_0)\cdot n] V\cdot W \,\mathrm{d}\gamma.$$

Now, since the test function W vanishes on Γ_D, the weak formulation reduces to

$$\int_\Omega \partial_t V\cdot W \,\mathrm{d}\omega + \int_\Omega [(V + V_0)\cdot \nabla(V + V_0)]\cdot W \,\mathrm{d}\omega - \int_\Omega \operatorname{div} \sigma(V, q)\cdot W \,\mathrm{d}\omega$$

$$+ \int_{\Gamma_N} [\![\sigma(V, q)\cdot n + \{\alpha[(V + V_0)\cdot n]^+ - \tfrac{1}{2}(V + V_0)\cdot n\} V]\!]\cdot W \,\mathrm{d}\gamma = \int_{\Gamma_N} H\cdot W \,\mathrm{d}\gamma,$$

$$\int_\Omega \pi \operatorname{div} V \,\mathrm{d}\omega = 0.$$

Thus V is the solution of the boundary value problem

$$\partial_t V + [(V + V_0)\cdot \nabla](V + V_0) - \operatorname{div} \sigma(V, q) = 0 \quad \text{in } \Omega \times (0, T),$$

$$\operatorname{div} V = 0 \quad \text{in } \Omega \times (0, T),$$

$$V(0) = U_0 - V_0 \quad \text{in } \Omega,$$

$$V = 0 \quad \text{on } \Gamma_D \times (0, T),$$

$$\sigma(V, q)\cdot n + \{\alpha[(V + V_0)\cdot n]^+ - \tfrac{1}{2}(V + V_0)\cdot n\} V = H \quad \text{on } \Gamma_N \times (0, T),$$

where α takes the value 0, $\frac{1}{2}$ and 1 respectively for the three forms of the Θ-function.

Finally, since (V_0, p_0) is the solution of the Stokes problem, we obtain

$$\partial_t U + (U\cdot \nabla)\cdot U - \operatorname{div} \sigma(U, p) = 0 \quad \text{in } \Omega \times (0, T),$$

$$\operatorname{div} U = 0 \quad \text{in } \Omega \times (0, T),$$

$$U(0) = U_0 \quad \text{in } \Omega,$$

$$U = U_D \quad \text{on } \Gamma_D \times (0, T),$$

$$\sigma(U, p)\cdot n - \tfrac{1}{2}\Theta(U\cdot n)(U - V_0) = \sigma(V_0, p_0)\cdot n + H \quad \text{on } \Gamma_N \times (0, T)$$

for some appropriate p_0. Thus U is the solution of (2)–(4) and (6) with $H = G_3 - \sigma(V_0, p_0) \cdot n$, where H is given by the physics.

We point out to the reader that for Stokes flow conditions (5) reduce to

$$\tilde{\sigma}(U, p) \cdot n = \tilde{G}_k \quad \text{on } \Gamma_N \times (0, T)$$

and conditions (6) reduce to

$$\sigma(U, p) \cdot n = G_k \quad \text{on } \Gamma_N \times (0, T),$$

and that in this last case the G_k must be non-zero because $\sigma(U, p).n = 0$ is not compatible with Poiseuille flow.

The condition of pseudotraction equal to zero is often used in the literature (see e.g. Reference 1, 9 and 10 and references cited therein), even for the Navier–Stokes equations, although we are not able to show that this leads to a well-posed problem. On the contrary, if we add inertial terms as in (5), we can get a well-posed problem. Here we consider both stress tensors and add some new non-linear terms for the Navier–Stokes model.

4. APPLICATION TO A CHANNEL

In this paper the boundary conditions above are tested in two dimensions to compute the incompressible flow behind a cylinder in a channel as shown in Figure 1. The numerical tests have shown that for the physical stress tensor the condition (6c) with $\Theta(a) = -a^-$ is the best one. Indeed, this condition is the only one that takes into account only the global flow U in the function Θ. Besides, the choice of $\Theta(a) = -a^-$ implies that when the flow is outgoing, the condition locally reduces to the Stokes flow condition, but when the flow is incoming, a traction term deduced from inertial terms is added to convey the vortices downstream.

We take a Poiseuille flow with a flow rate of unity upstream and a no-slip condition on the rigid boundaries as follows:

$$U = (6x_2(1 - x_2), 0) \quad \text{on } \Gamma_{D,1},$$

$$U = (0, 0) \quad \text{on } \Gamma_{D,0}.$$

Besides, on the artificial boundary Γ_N there is no external force given by the phsyics, so $H = 0$. This implies

$$G_3 = \sigma(V_0, p_0) \cdot n.$$

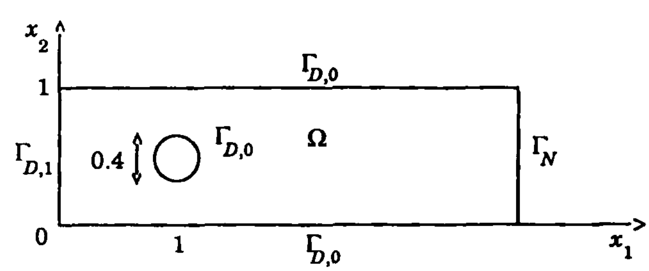

Figure 1. Domain and notations

When Γ_N is not too close to the obstacle, the solution of Stokes flow is a Poiseuille flow downstream. Thus with a flow rate of unity we can take

$$U_N = (6x_2(1 - x_2), 0),$$

$$\sigma(V_0, p_0) \cdot n = (0, 6(1 - 2x_2)).$$

Therefore condition (6c) reads for $U = (u_1, u_2)$ and p

$$\frac{2}{Re}\frac{\partial u_1}{\partial x_1} - p - \tfrac{1}{2}\Theta(u_1)[u_1 - 6x_2(1 - x_2)] = 0, \tag{7a}$$

$$\frac{1}{Re}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) - \tfrac{1}{2}\Theta(u_1)u_2 = \frac{6}{Re}(1 - 2x_2). \tag{7b}$$

This can be generalized by taking, instead of Poiseuille flow, U_N equal to the trace on Γ_N of a steady solution of the Navier–Stokes equations computed on a larger domain.

Let us notice that these boundary conditions link the three unknowns (u_1, u_2, p). Consequently they are very well adapted to a method that involves the pressure explicitly. Moreover, by adding the continuity equation to equations (7a) and (7b), we can derive uniquely the three unknowns.

In a mixed formulation, where the pressure is known when computing the velocity, equations (7a) and (7b) allow us to determine the two variables (u_1, u_2). On the contrary, in a strongly coupled approach we need explicitly the three unknowns downstream. Then we add the continuity equation to get one component of the velocity and the pressure is given by equation (7a).

For the three forms of the Θ-function we have three different conditions very close to each other. Taking Θ equal to the identity ($\alpha = 0$) corresponds to the natural Neumann-like condition and eliminates the non-linear boundary term in the weak formulation. For the other forms of $\Theta(\alpha \neq 0)$ we take into account the direction of the flow through the artificial boundary. Let us note that if $\Theta(a) = -a^-$ ($\alpha = \frac{1}{2}$), the inertial term vanishes when the flow is outgoing and so the boundary condition reduces to the condition of Stokes flow.

In this work we choose to present our conditions in a channel with solid walls. That is why the reference flow is a Poiseuille flow. In other respects, if we consider the flow around a cylinder in an open channel, we can take as reference flow the uniform flow $(u_1, u_2) = (1, 0)$. In this case the boundary conditions (7) become

$$\frac{2}{Re}\frac{\partial u_1}{\partial x_1} - p - \tfrac{1}{2}\Theta(u_1)(u_1 - 1) = 0,$$

$$\frac{1}{Re}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) - \tfrac{1}{2}\Theta(u_1)u_2 = 0.$$

Remark

Instead of conditions (7) we can also use for $\Theta(a) = -a^-$

$$\frac{2}{Re}\frac{\partial u_1}{\partial x_1} - p + \tfrac{1}{2}(u_1)^- u_1 = 0,$$

$$\frac{1}{Re}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) + \tfrac{1}{2}(u_1)^- u_2 = \frac{6}{Re}(1 - 2x_2).$$

The same condition for the pseudotraction gives

$$\frac{1}{Re}\frac{\partial u_1}{\partial x_1} - p + \tfrac{1}{2}(u_1)^- u_1 = 0,$$

$$\frac{1}{Re}\frac{\partial u_2}{\partial x_1} + \tfrac{1}{2}(u_1)^- u_2 = 0,$$

which is a condition that does not need a reference flow.

5. NUMERICAL RESULTS

The numerical tests are performed with condition (6c), with $\Theta(a) = -a^-$, for a large range of Reynolds numbers. For $Re = 100$ we observe a stable steady solution with a recirculation just behind the cylinder. In Figure 2 we show by the streamlines that the artificial boundary can cut the recirculation right in the middle without any significant perturbation. The broken shape of the cylinder is due to the approximation by square cells.

For higher Reynolds numbers the steady solution loses its stability to the benefit of a purely periodic solution. For $Re = 200$ the recirculation goes up and down across the symmetric axis of the domain and the flow does not present any other vortices. In particular, there are no bubbles travelling downstream. The cut shown in Figure 3 demonstrates clearly that the condition (6c) is also efficient for unsteady flow. However, we can see very small perturbations localized in the last cells downstream. These perturbations do not affect the whole flow and remain close to the artificial boundary even for a long-time simulation.

When the flow becomes more complex, strong vortices are convected downstream and cross the artificial boundary without any reflections, as shown in Figure 4. Small perturbations can

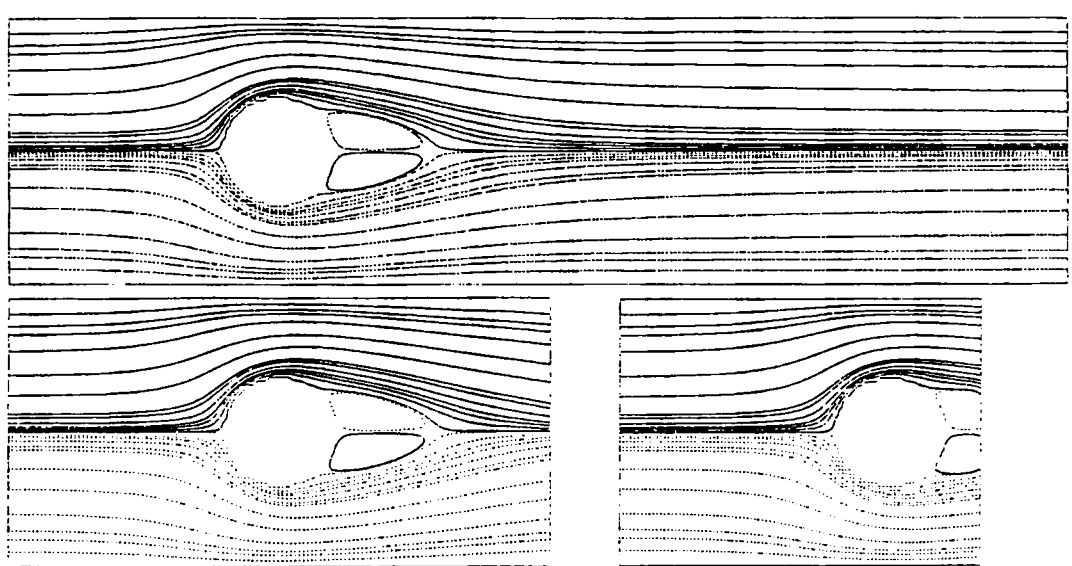

Figure 2. Streamlines at $Re = 100$

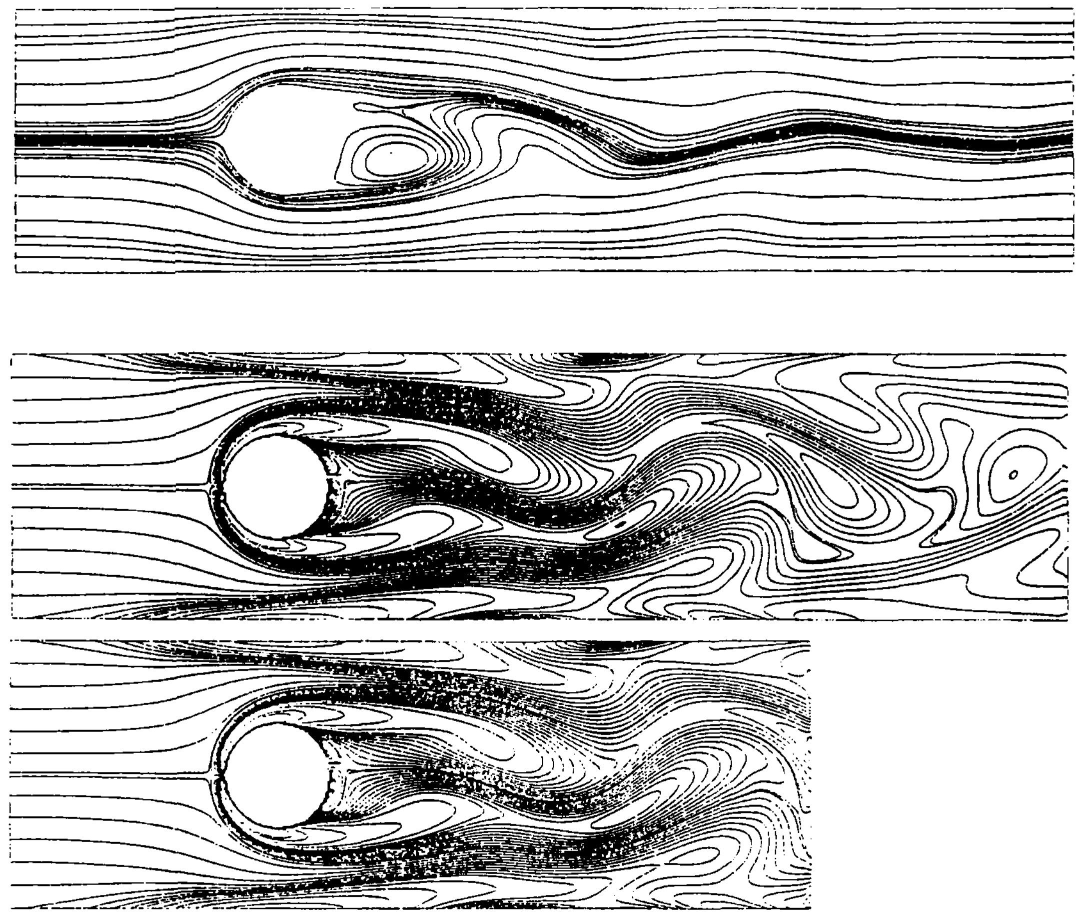

Figure 3. Solution at $Re = 200$: streamlines and vorticity lines

be seen in particular on the vorticity lines. These perturbations remain confined in the last cells and are more visible on the vorticity which is not a primary variable. They do not induce any reflections and have no action on the long-time behaviour of the flow. Indeed, we do not observe any delay for solutions computed on a truncated domain. Moreover, the spectrum of the solution does not change with the position of the cut.

For the above flows the Stokes condition

$$\tilde{\sigma}(U, p)\cdot n = 0 \quad \text{or} \quad \sigma(U, p)\cdot n = \sigma(V_0, p_0)\cdot n \tag{8}$$

gives about the same results as condition (6c). However, for higher Reynolds numbers these conditions induce strong reflections (Figure 5) and it is necessary to take into account the non-linear terms to avoid numerical blow-up.

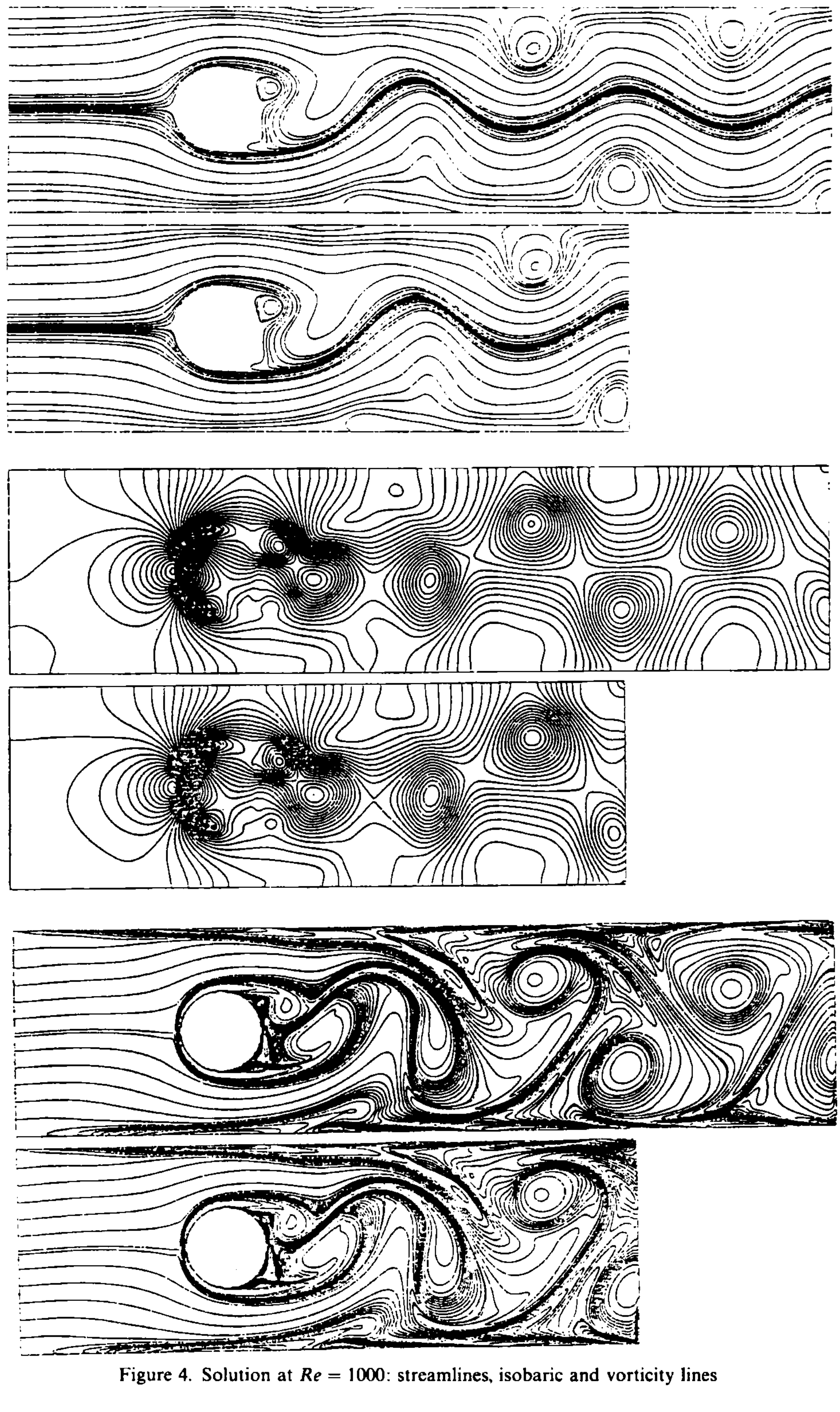

Figure 4. Solution at $Re = 1000$: streamlines, isobaric and vorticity lines

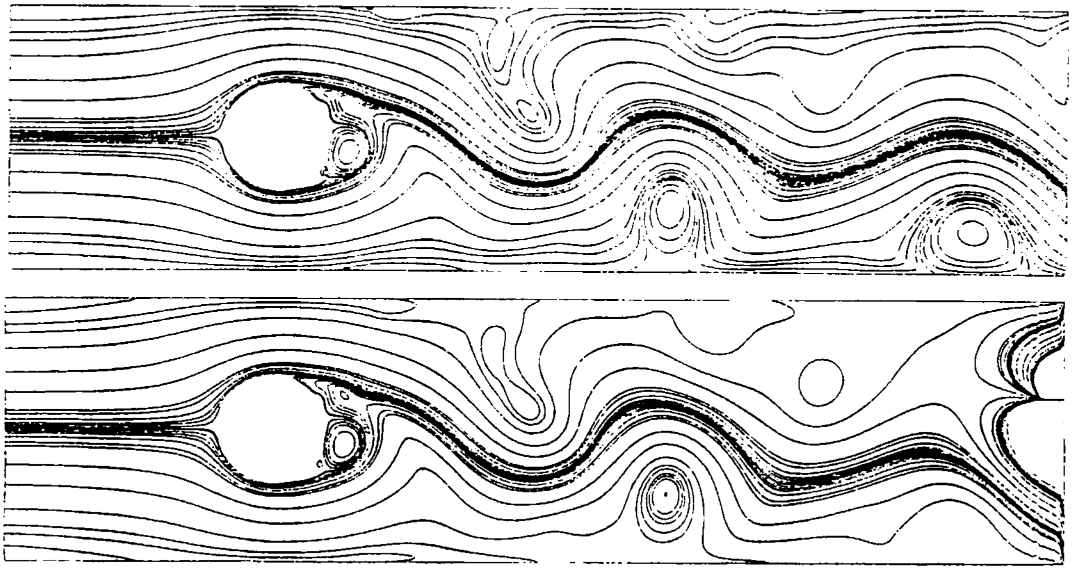

Figure 5. Solution at Re = 10,000 with conditions (6c) (top) and (8) (bottom)

6. COMMENTS ON BOUNDARY CONDITIONS FOR THE CHANNEL

We do not intend to give an exhaustive review of the boundary conditions for the Navier–Stokes equations, but only to make some comments about those which are the closest to ours. We begin with the conditions proposed by Peyret and Rebourcet:[2]

$$\frac{\partial p}{\partial x_2} = -\frac{\partial u_2}{\partial t} - u_2 \frac{\partial u_2}{\partial x_2} + \frac{1}{Re}\frac{\partial^2 u_2}{\partial x_2^2} \quad \text{on } \Gamma_N, \tag{9a}$$

$$\frac{\partial u_2}{\partial x_1} = 0 \quad \text{on } \Gamma_N, \tag{9b}$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 \quad \text{on } \Gamma_N. \tag{9c}$$

The first equation comes from a parabolization of the Navier–Stokes equations. It gives the pressure by integration along Γ_N. The second equation is a usual condition on the pseudotraction $\tilde{\sigma}(U, p)$. Here the second component is set equal to zero. The third equation is the continuity equation. These boundary conditions are stable and well adapted to steady laminar flow, but give reflections when strong vortices cross the artificial boundary.

Several authors[1,9,10] use $\tilde{\sigma}(U, p)\cdot n = 0$ on Γ_N for Stokes flow. Here we also propose $\sigma(U, p)\cdot n = \sigma(V_0, p_0)\cdot n$, where (V_0, p_0) is a reference flow. Although these conditions do not take into account the convection terms, numerical experiments show that both conditions can be used for the Navier–Stokes equations except for high Reynolds numbers.

In the same way Verfürth[11,12] adds a penalty boundary term on the normal component of the traction in a mixed formulation coupled to $U \cdot n = g$ on Γ_N. We point out that $\sigma(U, p)\cdot n \cdot n = 0$ on Γ_N makes sense for Poiseuille flow, but not $\sigma(U, p)\cdot n \cdot \tau = 0$.

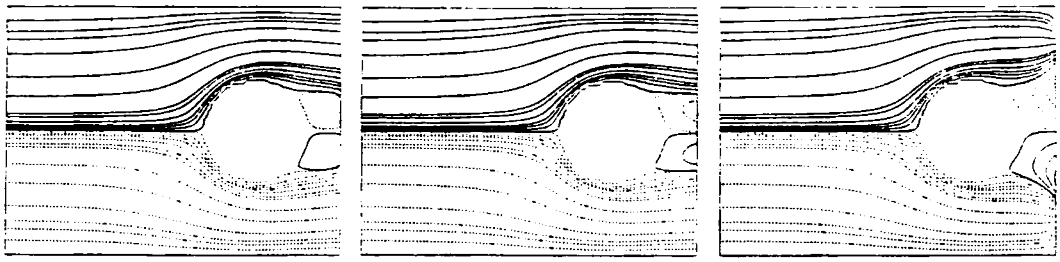

Figure 6. Cut of the domain for $Re = 100$ with conditions (7) (left), (9) (middle) and (10) (right)

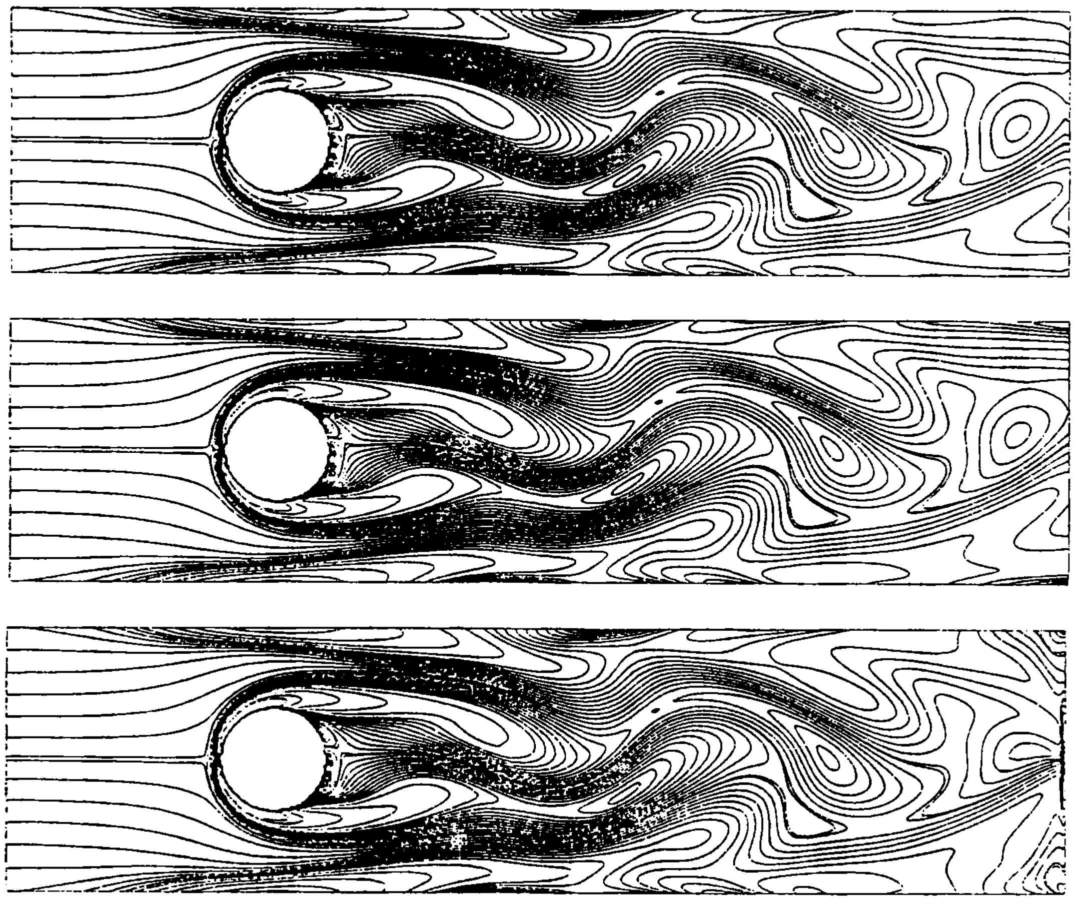

Figure 7. Solution at $Re = 200$ with conditions (7) (top), (9) (middle) and (10) (bottom)

Finally, Bègue *et al.*[4,5] establish for the Stokes problem numerous conditions on the pressure. In particular, the boundary equation $p + \frac{1}{2}(u_1^2 + u_2^2) = p_0$ on Γ_N links the pressure to the modulus of the velocity. In another work Pironneau[3] substitutes for $\sigma(U, p) \cdot n = 0$ on Γ_N the conditions

$$u_1 = g_1, \qquad \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = g_2, \qquad \frac{\partial p}{\partial x_1} = g_3$$

Figure 8. Solution at $Re = 1000$ with conditions (9) before numerical blow-up

to get a well-posed Stokes problem.

We perform numerical experiments with conditions (9) and the conditions

$$p = -\tfrac{1}{2}(u_1^2 + u_2^2) \quad \text{on } \Gamma_N, \tag{10a}$$

$$\frac{\partial u_2}{\partial x_1} = 0 \quad \text{on } \Gamma_N, \tag{10b}$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 \quad \text{on } \Gamma_N \tag{10c}$$

to test another condition on the pressure.

For low Reynolds number, the conditions (9) gives much the same solution as ours with slight discrepancies at the boundary (see Figures 6 and 7). It is smoother in the middle but shows some extra gradients at the corners. On the other hand, conditions (10) pertub the solution in a significant region downstream, as shown in Figures 6 and 7. At higher Reynolds numbers, for which strong vortices cross the artificial boundary, we are not able to get the solution with both conditions (9) and (10) owing to strong reflections downstream, as shown in Figure 8.

7. CONCLUSIONS

In this work we propose new non-linear boundary conditions on an open boundary to compute the solution of the unsteady incompressible Navier–Stokes equations. We establish Neumann-like boundary conditions associated with a weak formulation which yields a well-posed problem.

The numerical tests presented in this paper prove the accuracy and robustness of these conditions. Indeed, we can cut the domain everywhere without significantly perturbing the flow, even when strong vortices cross the artificial boundary. Moreover, it appears that it is necessary to take into account the non-linear terms to compute chaotic flows.

REFERENCES

1. P. M. Gresho, 'Incompressible fluid dynamics: some fundamental formulation issues', *Ann. Rev. Fluid Mech.*, **23**, 413–453 (1991).
2. R. Peyret and B. Rebourcet, 'Développement de jets en fluides stratifiés', *J. Méc. Théor. Appl.*, **1**, 467–491 (1982).
3. O. Pironneau, 'Conditions aux limites sur la pression pour les équations de Stokes et de Navier–Stokes', *C.R. Acad. Sci. Paris, Sér. I*, **303**, 403–406 (1986).
4. C. Bègue, C. Conca, F. Murat and O. Pironneau, 'A nouveau sur les équations de Stokes et de Navier–Stokes avec des conditions aux limites sur la pression,' *C.R. Acad. Sci. Paris, Sér. I*, **304**, 23–28 (1987).
5. C. Bègue, C. Conca, F. Murat and O. Pironneau, 'Les équations de Stokes et de Navier–Stokes avec des conditions aux limites sur la pression,' in H. Brezis and J. L. Lions (eds), *Pitman Research Notes in Mathematics Series*, 181, *Nonlinear Partial Differential Equations and Their Applications, Collège de France Seminar IX*, Pitman, London, 1988, pp. 179–264.
6. L. Halpern, 'Artificial boundary conditions for the linear advection diffusion equation', *Math. Comput.*, **46**, 425–438 (1986).
7. L. Halpern and M. Schatzman, 'Artificial boundary conditions for incompressible viscous flows', *SIAM J. Math. Anal.*, **20**, 308–353 (1989).
8. C. Conca, 'Approximation de quelques problémes de type Stokes par une méthode d'éléments finis mixtes,' *Numer. Math.*, **45**, 75–91 (1984).
9. R. K. Ganesh, 'Evaluation of the drag force by integrating the energy dissipation rate in Stokes flow for 2D domains using the FEM,' *Int. j. numer. methods fluids*, **13**, 557–578 (1991).
10. P. M. Gresho, 'On the theory of semi-implicit projection methods for viscous incompressible flow and its implementation via a finite element method that also introduces a nearly consistent mass matrix. Part 1: Theory', *Int. j. numer. methods fluids*, **11**, 587–620 (1990).
11. R. Verfürth, 'Finite element approximation of incompressible Navier–Stokes equations with slip boundary condition', *Numer. Math.*, **50**, 697–721 (1987).
12. R. Verfürth, 'Finite element approximation of incompressible Navier–Stokes equations with slip boundary condition II', *Numer, Math.*, **59**, 615–636 (1991).
13. Ch.-H. Bruneau and P. Fabrie, 'New efficient boundary conditions for incompressible Navier–Stokes equations: a well-posedness result,' submitted.